Twist Positivity for Lagrangian Symmetries*

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Abstract. We prove twist positivity and positivity of the pair correlation function for combined spatial and internal symmetries of free bosonic Lagrangians. We work in a general setting, extending the results obtained in *Twist Positivity* [1].

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1 Introduction

In this paper we generalize the results of *Twist Positivity* [1] to a wide class of symmetries. We investigate the (unitary) implementation U_S of a symmetry S of a classical, free field Lagrangian. We establish twist positivity of the the partition function $Z_{\beta,U}$ twisted by the symmetry U_S, namely $Z_{\beta,U} > 0$, and show positivity of the pair correlation operator that is twisted by U_S, namely $C_\beta > 0$. After considering a simple example of a free field in §1.1, we give the general definitions of these concepts in §1.2–1.3. The methods here are applicable both in quantum field theory and also in related problems in statistical physics.

1.1 Free Bosonic Fields on Compact Manifolds.

Let M be a compact Riemannian manifold, $E \to M$ a Hermitian vector bundle on M endowed with a compatible connection, and let Δ_E denote the corresponding (positive self-adjoint) Laplacian on the Hilbert space $L^2(E)$ of square integrable sections of E. Let $\langle \cdot, \cdot \rangle$ denote the inner-product[1] on $L^2(E)$, and let $\mathcal{D}(\triangle_E)$ be the domain of $\triangle_E$.

The corresponding free field theory has the Lagrangian $\mathcal{L} : \mathcal{D}\left(\triangle_E^{1/2}\right) \times L^2(E) \to \mathbb{R}$, defined by,

$$\mathcal{L}\left(\varphi_{\rm cl}, \frac{\partial \varphi_{\rm cl}}{\partial t}\right) = \left\langle \frac{\partial \varphi_{\rm cl}}{\partial t}, \frac{\partial \varphi_{\rm cl}}{\partial t} \right\rangle - \left\langle \triangle_E^{1/2} \varphi_{\rm cl}, \triangle_E^{1/2} \varphi_{\rm cl} \right\rangle - m^2 \left\langle \varphi_{\rm cl}, \varphi_{\rm cl} \right\rangle , \tag{1}$$

where $m \geq 0$ is the *mass* of the field. The characteristic feature of free bosonic quantum field theory is that the time evolution is prescribed by a *linear* partial differential equation of second order. The dynamics corresponding to this Lagrangian via the Euler variational principle reads,

$$\left(\frac{\partial^2}{\partial t^2} + \triangle_E + m^2\right) \varphi_{\rm cl}(t,x) = 0 \ . \tag{2}$$

Since the manifold M is compact, the self-adjoint operator Δ_E has discrete spectrum and there is an orthonormal basis[2] of $L^2(E)$, $\{\varphi_{{\rm cl},k}\}_{k\in K}$, consisting of eigensections of Δ_E,

$$\left(\triangle_E + m^2\right) \varphi_{{\rm cl},k} = \omega_k^2 \varphi_{{\rm cl},k} \ . \tag{3}$$

Restricting consideration to the case that all the ω_k^2 are strictly positive, each solution to eq.(2) may be expanded as

$$\varphi_{\rm cl}(t,x) = \sum_{k\in K} \frac{1}{\sqrt{2\omega_k}} (\bar{\alpha}_+(k) e^{i\omega_k t} + \alpha_-(-k) e^{-i\omega_k t}) \varphi_{{\rm cl},k}(x) \ , \tag{4}$$

where $\alpha_\pm(\pm k)$ are complex coefficients. Canonical quantization of that system replaces the complex coefficients $\alpha_\pm(\pm k)$ by operators, also denoted by $\alpha_\pm(\pm k)$, satisfying the canonical commutation relations,

$$\begin{aligned} \left[\alpha_\pm(\pm k), \alpha_\pm^*(\pm k')\right] &= \delta_{k,k'} \\ \left[\alpha_\pm(\pm k), \alpha_\pm(\pm k')\right] &= \left[\alpha_+(k), \alpha_-(-k')\right] = \left[\alpha_+(k), \alpha_-^*(-k')\right] = 0 \ . \end{aligned} \tag{5}$$

[1] Inner products are antilinear in the first argument.

[2] See §2.2 for a basis-independent quantization.

The $\alpha_{\pm}^{*}(\pm k)$ act on a Fock space $\mathfrak{F}$, which is the Hilbert space spanned by all vectors of the form $\alpha_{\pm}^{*}(\pm k_1)...\alpha_{\pm}^{*}(\pm k_n)|0\rangle$, where the unit vector $|0\rangle$ (called the *vacuum*) is in the nullspace of all the $\alpha_{\pm}(\pm k)$ and in the domain of any product of $\alpha_{\pm}^{*}(\pm k)$'s. The charged "one-particle" subspaces of $\mathfrak{F}$, which are the spans of $\left\{\alpha_{-}^{*}(-k)|0\rangle\right\}_{k\in K}$ and $\left\{\alpha_{+}^{*}(k)|0\rangle\right\}_{k\in K}$, play a special role in the theory. Natural *linear* isomorphisms between these spaces and $L^2(E)$ and its dual, respectively, will be exploited in §3.1.

1.2 Bosonic Quantization of Admissible Quadratic Lagrangians

We work here with a generalization of the fields in §1.1.

Assumptions:

1. Replace the space $L^2(E)$ by a separable complex Hilbert space $\mathcal{E}$, called the **space of classical fields**. The canonical pairing between $\mathcal{E}$ and its dual $\mathcal{E}^*$ is denoted by $(\cdot,\cdot):\mathcal{E}^*\times\mathcal{E}\to\mathbb{C}$.

2. Replace the operator $\sqrt{m^2+\triangle_E}$ by a positive self-adjoint **classical frequency operator** $\Omega:\mathcal{E}\to\mathcal{E}$ which is bounded below by a positive constant $\lambda>0$. The **ground state energy** $\mu>0$ is the largest such λ.

3. The compactness of M is replaced by the assumption that Ω is **Θ-summable**, i.e. that $\operatorname{Tr}e^{-\beta\Omega}<\infty$ for all $\beta>0$.

4. The **free Lagrangian** $\mathfrak{L}:\mathcal{D}(\Omega)\times\Omega\to\mathbb{R}$ is given by

$$\mathfrak{L}(\varphi,d\varphi/dt)=\langle d\varphi/dt,d\varphi/dt\rangle-\langle\Omega\varphi,\Omega\varphi\rangle. \tag{6}$$

Note that condition (2) rules out the case of a massless ($m=0$) scalar field on the circle S^1 parametrized by $\theta\in[0,2\pi)$ unless the Laplacian $\triangle_{S^1}=-d^2/d\theta^2$ is *twisted*. For ρ not a multiple of 2π, the *twisted Laplacian* $\triangle_{S^1}^{\rho}$ is the self-adjoint extension of $-d^2/d\theta^2$ acting on smooth functions on S^1 which satisfy

$$\lim_{\theta\to 2\pi^-}\frac{d^n f}{d\theta^n}=e^{i\rho}\frac{d^n f}{d\theta^n}$$

for $n\in\mathbb{N}$.

Definition 1 *An Ω satisfying the strict positivity and Θ-summability assumptions (2-3) is called* **admissible**, *as is its associated free Lagrangian. The* ***canonical antilinear isomorphism*** *$f\mapsto\bar{f}:\mathcal{E}\to\mathcal{E}^*$ is given by*

$$\left(\bar{f},g\right)=\langle f,g\rangle$$

for all $g\in\mathcal{E}$. Given a linear or antilinear operator $A:\mathcal{E}\to\mathcal{E}$, the ***conjugate transformation*** *$\bar{A}:\mathcal{E}^*\to\mathcal{E}^*$ is given by*

$$\bar{A}\bar{g}=\overline{Ag}.$$

To quantize this theory in the usual way, let $\{e_k\}_{k\in K}$ denote an orthonormal basis of $\mathcal{E}$ consisting of eigenvectors of Ω, namely,

$$\Omega e_k = \omega_k e_k \ , \quad \text{for all } k \in K \ . \tag{7}$$

Let $Op(\mathfrak{F})$ denote the set of linear operators on $\mathfrak{F}$. The *real-time* quantum field $\varphi_{RT} : \mathbb{R} \times \mathcal{E}^* \to Op(\mathfrak{F})$ is the operator-valued function defined by

$$\varphi_{RT}\left(t, \bar{f}\right) = \sum_{k\in K} \frac{1}{\sqrt{2\omega_k}} \left(\alpha_+^*(k)\, e^{i\omega_k t} + \alpha_-(-k)\, e^{-i\omega_k t}\right) \left(\bar{f}, e_k\right).$$

The operators $\alpha_\pm(\pm k)$ are required to satisfy the canonical commutation relations (5), and to act on Fock space $\mathfrak{F}$, which is isomorphic to the symmetric tensor algebra $\exp_{\otimes_S} \mathcal{E}\oplus\mathcal{E}^*$.[3] The *Hamiltonian* H and *particle number operator* N of the system are defined as

$$H = \sum_{k\in K} \omega_k \left(\alpha_+^*(k)\alpha_+(k) + \alpha_-^*(-k)\alpha_-(-k)\right)$$
$$N = \sum_{k\in K} \left(\alpha_+^*(k)\,\alpha_+(k) + \alpha_-^*(-k)\,\alpha_-(-k)\right).$$

The fields $\varphi_{RT}\left(t, \bar{f}\right)$ and $\varphi_{RT}^*(t, f)$ commute and $\varphi_{RT}\left(t, \bar{f}\right)(N+1)^{-1/2}$ is bounded. We then infer that the closure of $\varphi_{RT}\left(t, \bar{f}\right)$ defined on the domain $\mathfrak{F}_0$ that is algebraic subspace spanned by states of finite particle number, is normal. This follows by an application of Nelson's analytic vector theorem, Lemma 5.1 of [2], since the vectors in $\mathfrak{F}_0$ are a common set of analytic vectors for the real and imaginary parts of the field. Note that for $\bar{f} \in \mathcal{D}\left(\bar{\Omega}^2\right)$, the field φ_{RT} satisfies the Klein-Gordon equation

$$\frac{\partial^2}{\partial t^2}\varphi_{RT}\left(t, \bar{f}\right) + \varphi_{RT}\left(t, \bar{\Omega}^2 \bar{f}\right) = 0, \tag{8}$$

where the derivative is taken strongly on $\mathfrak{F}_0$.

The conjugate field $\varphi_{RT}^* : \mathbb{R} \times \mathcal{E} \to Op(\mathfrak{F})$ is given by

$$\varphi_{RT}^*(t, f) = \left(\varphi_{RT}\left(t, \bar{f}\right)\right)^* = \sum_{k\in K} \frac{1}{\sqrt{2\omega_k}} \left(\alpha_-^*(-k)\, e^{i\omega_k t} + \alpha_+(k)\, e^{-i\omega_k t}\right) (\bar{e}_k, f)$$

The *imaginary-time* fields $\varphi : [0,\infty) \times \mathcal{E}^* \to Op(\mathfrak{F})$ and $\bar{\varphi} : [0,\infty) \times \mathcal{E} \to Op(\mathfrak{F})$ are given by

$$\varphi\left(t, \bar{f}\right) = e^{-tH} \varphi_{RT}\left(0, \bar{f}\right) e^{tH} \quad , \quad \bar{\varphi}(t, f) = e^{-tH} \varphi_{RT}^*(0, f)\, e^{tH}$$

Again, $\varphi\left(t, \bar{f}\right)$ and $\bar{\varphi}(t, f)$ give well-defined normal operators with core $\mathfrak{F}_0$ when $t \geq 0$.

[3] We use α instead of the more standard variable a to remind the reader that the symbol k need not correspond to momentum. The notation $\alpha_-(-k)$ is preferred over $\alpha_-(k)$ to resemble the standard quantization of the free complex scalar field in a rectangular box. In that case, the basis of $\mathcal{E}$ can be chosen to be of the form $e_k = e^{-ikx}$, where k ranges over some lattice. Then $(\bar{e}_k, f) = \int e_{-k}(x)\, f(x)\, dx$ and $\bar{\alpha}_-(-k) = a_-(-k)$.

1.3 Twist Positivity and the Twisted Pair Correlation Function

The partition function of the system is defined as

$$Z_\beta = \mathrm{Tr}\left(e^{-\beta H}\right) \ , \tag{9}$$

so one needs the heat operator to be trace class. This is a consequence of the admissibility of Ω.

Definition 2 *For a unitary operator $U : \mathfrak{F} \to \mathfrak{F}$ commuting with the Hamiltonian H, the **partition function twisted by U** is*

$$Z_{\beta,U} = \mathrm{Tr}\left(Ue^{-\beta H}\right).$$

*We say that U is a **twist positive** with respect to H if*

$$Z_{\beta,U} > 0$$

for all $\beta > 0$.

It was observed in [1] that, surprisingly enough, many interesting symmetries U are twist positivity, both in particle quantum theory and in quantum field theory. In this work, we generalize previous results by considering the following natural class of symmetries:

Definition 3 *A bounded linear or antilinear operator $S : \mathcal{E} \to \mathcal{E}$ is a **Lagrangian symmetry**, if S restricts to a bijection of $\mathfrak{D}(\Omega)$ onto itself and*

$$\mathcal{L}(\varphi, \dot{\varphi}) = \mathcal{L}(S\varphi, S\dot{\varphi}) \ . \tag{10}$$

Hence S is a Lagrangian symmetry iff it preserves the closed quadratic forms

$$\varphi \mapsto \mathcal{L}(\varphi, 0) \quad \text{and} \quad d\varphi/dt \mapsto \mathcal{L}(0, d\varphi/dt)$$

and their domains. From the one-to-one correspondence between closed positive quadratic forms and positive self-adjoint operators (see [3]), it follows that *S is a Lagrangian symmetry of $\mathcal{L}$ iff $[S, \Omega] = 0$ and S is unitary or anitunitary.*

For each Lagrangian symmetry S, we shall denote by U_S its implementation on Fock space $\mathfrak{F}$ (see §3). The U_S have a characteristic action on $\mathfrak{F}$. Indeed, the set of such implementations is precisely the set of *unitary*[4] operators $U : \mathfrak{F} \to \mathfrak{F}$ such that

1. U commutes with the Hamiltonian H, the number operator N, and with the combined time-charge reversal operator TC, which is constructed in Theorem 11 below.
2. U preserves the vacuum, and U acts independently on each particle in a multiparticle state, unaffected by the presense of other particles.
3. U either sends particles to particles of the same charge (for S unitary) or to particles of opposite charge (for S antiunitary).

That properties 1-3 hold for implementations of Lagrangian symmetries is proven in lemmas 10 and 12. Theorem 13 implies that all such symmetries U are implementations of Lagrangian symmetries.

We now give our main results.

[4]Note that an antiunitary Lagrangian symmetry will have a unitary implementation on $\mathfrak{F}$.

Theorem 4 *The Fock space implementation U_S of a Lagrangian symmetry S of an admissible free Lagrangian is twist positive.*

We define the twisted pair correlation function and associated objects:

Definition 5 *Let $0 \leq t, s \leq \beta$, $\bar{f} \in \mathcal{E}^*$ and $g \in \mathcal{E}$. The* **time-ordered product** *is given by*

$$(\varphi(t, \bar{f})\bar{\varphi}(s, g))_+ = \begin{cases} \varphi(t, \bar{f})\bar{\varphi}(s, g) & \text{if } t < s \\ \bar{\varphi}(s, g)\varphi(t, \bar{f}) & \text{if } t \geq s \end{cases} \tag{11}$$

For a unitary Lagrangian symmetry S, the **twisted pair correlation function $\mathbf{C}_{\beta,\mathbf{U_S}}$** *is*

$$C_{\beta,U_S}(t, \bar{f}; s, g) = \frac{1}{Z_{\beta,U_S}} \mathrm{Tr}\left((\varphi(t, \bar{f})\bar{\varphi}(s, g))_+ U_S e^{-\beta H}\right) \ . \tag{12}$$

The twisted pair correlation function is the integral kernel of the **twisted pair correlation operator $\mathbf{C}_\beta$** *on the* **path space** $\mathcal{T}_\beta = L^2\left([0, \beta); \mathcal{E}\right) \cong L^2\left([0, \beta)\right) \otimes \mathcal{E}$ *of square-integrable functions from $[0, \beta)$ to $\mathcal{E}$. The operator C_β is that which satisfies*

$$\left\langle \tilde{f}, C_\beta \tilde{g} \right\rangle_{\mathcal{T}_\beta} = \int_0^\beta \int_0^\beta C_{\beta,U_S}\left(t, \overline{\tilde{f}(t)}; s, \tilde{g}(t)\right) ds\, dt, \tag{13}$$

where $\tilde{f}, \tilde{g} \in \mathcal{T}_\beta$. Define the **twisted derivative D** *as i times the self-adjoint extension of $\frac{1}{i}\frac{\partial}{\partial t}$ acting on smooth functions $\widetilde{g} : [0, \beta) \to \mathcal{E}$ such that for all $n \in \mathbb{N}$,*

$$\frac{\partial^n \tilde{g}}{\partial t^n}(0) = \lim_{t \to \beta^-} S^* \frac{\partial^n \tilde{g}}{\partial t^n}(t) \ . \tag{14}$$

We prove in §5 the following

Theorem 6 *If S is a unitary Lagrangian symmetry of an admissible free Lagrangian then*

$$C_\beta = \left(-D^2 + \Omega^2\right)^{-1}, \tag{15}$$

where Ω is identified with $I \otimes \Omega$.

In §6, we modify and extend this theorem to antiunitary symmetries and to the case of real scalar fields.

The positivity of the operator C_β ensures the existence of a countably additive Borel measure whose moments are the twisted correlation functions of the free field. It would be interesting to extend the techniques of constructive quantum field theory to non-linear perturbations of the free theories we consider here.

2 Bosonic Quantization of Complex Free Fields

2.1 The standard $L^2(X)$ representation of $\mathcal{E}$.

Making contact with the standard physics notation for complex free Bosonic fields, we represent the space $\mathcal{E}$ as $L^2(X)$, for some measure space X, so that expressions involving fields $\varphi \in \mathcal{E}$ may be written in a familiar form as $\varphi(x)$, $x \in X$. We note that if one identifies a function $f \in L^2(X)$ with the linear functional on $L^2(X)$ given by

$$g \mapsto \int f(x)\, g(x)\, dx\,,$$

then the canonical isomorphism $\bar{\cdot}$ is just complex conjugation, and the conjugate transformation of A is given by

$$\bar{A}f = \overline{\left(A\bar{f}\right)},$$

where the bars on the right-side denote complex conjugation. Elements of $\mathcal{E}^*$ will always be represented below as $\bar{f}$, for some element $f \in \mathcal{E}$. In particular, we make no essential use of complex conjugation on $L^2(X)$, which is not a natural representation-independent operation on $\mathcal{E}$.[5]

The operator-valued linear functionals φ_{RT} and φ^*_{RT} are commonly expressed in suggestive notation as

$$\varphi_{RT}\left(t, \bar{f}\right) = \int \varphi_{RT}(t,x)\, \bar{f}(x)\, dx \tag{16}$$

and

$$\varphi^*_{RT}(t, f) = \int \varphi^*_{RT}(t,x)\, f(x)\, dx. \tag{17}$$

Here $\varphi_{RT}(t,x)$ and $\varphi^*_{RT}(t,x)$ are notational devices expressed as

$$\varphi_{RT}(t,x) = \sum_k \frac{1}{\sqrt{2\omega_k}} \left(e^{i\omega_k t}\alpha^*_+(k) + e^{-i\omega_k t}\alpha_-(-k)\right) e_k(x)$$

$$\varphi^*_{RT}(t,x) = \sum_k \frac{1}{\sqrt{2\omega_k}} \left(e^{i\omega_k t}\alpha^*_-(-k) + e^{-i\omega_k t}\alpha_+(k)\right) \bar{e}_k(x)\,,$$

where the summations are understood to be interchanged with the integrals in $(16-17)$. Note that $\varphi_{RT}(t,x)$ and $\varphi^*_{RT}(t,x)$ are not functions, and pointwise they are merely symbolic expressions.

As usual, the canonical commutation relations for the operators $\alpha_\pm$ are equivalent to the equal-time canonical commutators:

$$\left[\varphi_{RT}\left(t, \bar{f}\right), \frac{\partial \varphi^*_{RT}}{\partial t}(t,g)\right] = i\left(\bar{f}, g\right) = i\int \bar{f}(x)\, g(x)\, dx = i\langle f, g\rangle \tag{18}$$

$$\left[\varphi_{RT}\left(t, \bar{f}\right), \frac{\partial \varphi_{RT}}{\partial t}(t,\bar{g})\right] = \left[\varphi_{RT}\left(t, \bar{f}\right), \varphi^*_{RT}(t,g)\right] = \left[\varphi_{RT}\left(t, \bar{f}\right), \varphi_{RT}(t,\bar{g})\right] = 0. \tag{19}$$

[5] An example of a complex classical field theory given on an L^2 space with physically unnatural conjugation is the complex scalar field on the "twisted circle." Set $\mathcal{E} = L^2\left(S^1\right)$, $\Omega = \triangle^\rho_{S^1}$, where $\triangle^\rho_{S^1}$ is the twisted Laplacian defined in §1.1. Hence unless ρ is a multiple of π, we see that the L^2-conjugation on $\mathcal{E}$ is does not commute with least-action time evolution.

Here $\partial\varphi_{RT}/\partial t$ and $\partial\varphi_{RT}^*/\partial t$ are defined using the strong limit on $\mathfrak{F}_0$.

The Klein-Gordon equation (8) for $\varphi_{\rm cl}(t,x)$ is denoted by

$$\left(\partial_t^2+\Omega_x^2\right)\varphi_{\rm cl}(t,x)=0, \tag{20}$$

and the *conjugate* Klein-Gordon equation for $\bar{\varphi}_{\rm cl}(t,x)$ is

$$\left(\partial_t^2+\bar{\Omega}_x^2\right)\bar{\varphi}_{\rm cl}(t,x)=0. \tag{21}$$

Notice that when $\mathcal{E}$ is the space $L^2(E)$ of square integrable sections of the vector bundle E then $\bar{\varphi}_{\rm cl}$ is a section of the dual bundle E^*. This is why we refrained from identifying the space $\mathcal{E}$ with its dual.

2.2 Creation/Annihilation Functionals & Basis-Free Quantization

We introduce the creation and annihilation functionals, which play a role in the Fock space implementation of classical symmetries. We show that they are natural objects which come from a basis-independent method of quantization.

Definition 7 *The **linear creation functionals** $A_+^*:\mathcal{E}^*\to Op(\mathfrak{F})$ and $A_-^*:\mathcal{E}\to Op(\mathfrak{F})$ are defined by*

$$A_+^*\left(\bar{f}\right)=\sum_k\left(\bar{f},e_k\right)\alpha_+^*(k)=\sum_k\langle f,e_k\rangle_{\mathcal{E}}\,\alpha_+^*(k),\qquad \bar{f}\in\mathcal{E}^*$$
$$A_-^*(f)=\sum_k(\bar{e}_k,f)\,\alpha_-^*(-k)=\sum_k\langle e_k,f\rangle_{\mathcal{E}}\,\alpha_-^*(-k),\quad f\in\mathcal{E}.$$

*The linear creation functionals are well-defined operators on $\mathfrak{F}_0$. The **linear annihilation functionals** are given by*

$$A_+(f)=\left(A_+^*\left(\bar{f}\right)\right)^*=\sum_k(\bar{e}_k,f)\,\alpha_+(k)=\sum_k\langle e_k,f\rangle_{\mathcal{E}}\,\alpha_+(k)$$
$$A_-\left(\bar{f}\right)=\left(A_-^*(f)\right)^*=\sum_k\left(\bar{f},e_k\right)\alpha_-(-k)=\sum_k\langle f,e_k\rangle_{\mathcal{E}}\,\alpha_-(-k).$$

We state without proof the following

Theorem 8 *The creation and annihilation functionals satisfy for all $f,g\in\mathcal{E}$*

$$\left[A_+(f),A_+^*(\bar{g})\right]=(\bar{g},f)=\langle g,f\rangle_{\mathcal{E}} \tag{22}$$
$$\left[A_-\left(\bar{f}\right),A_-^*(g)\right]=\left(\bar{f},g\right)=\langle f,g\rangle_{\mathcal{E}}\ . \tag{23}$$

The dynamics of the $A_\pm^*$ are given by

$$e^{itH}A_+^*\left(\bar{f}\right)e^{-itH}=A_+^*\left(e^{it\bar{\Omega}}\bar{f}\right) \tag{24}$$
$$e^{itH}A_-^*(f)\,e^{-itH}=A_-^*\left(e^{it\Omega}f\right)\ . \tag{25}$$

Furthermore[6]

$$\varphi_{RT}\left(t,\bar{f}\right)=\frac{1}{\sqrt{2}}\left[A_{+}^{*}\left(\bar{\Omega}^{-1/2}e^{it\bar{\Omega}}\bar{f}\right)+A_{-}\left(\bar{\Omega}^{-1/2}e^{-it\bar{\Omega}}\bar{f}\right)\right] \tag{26}$$

$$\varphi_{RT}^{*}\left(t,f\right)=\frac{1}{\sqrt{2}}\left[A_{-}^{*}\left(\Omega^{-1/2}e^{it\Omega}f\right)+A_{+}\left(\Omega^{-1/2}e^{-it\Omega}f\right)\right], \tag{27}$$

and for $f\in\mathcal{D}\left(e^{t\Omega}\right)$

$$\varphi\left(t,\bar{f}\right)=\frac{1}{\sqrt{2}}\left[A_{+}^{*}\left(\bar{\Omega}^{-1/2}e^{-t\bar{\Omega}}\bar{f}\right)+A_{-}\left(\bar{\Omega}^{-1/2}e^{t\bar{\Omega}}\bar{f}\right)\right] \tag{28}$$

$$\bar{\varphi}\left(t,f\right)=\frac{1}{\sqrt{2}}\left[A_{-}^{*}\left(\Omega^{-1/2}e^{-t\Omega}f\right)+A_{+}\left(\Omega^{-1/2}e^{t\Omega}f\right)\right]. \tag{29}$$

If we define the maps

$$\begin{aligned}\Gamma_{+}:\mathcal{E}^{*}&\rightarrow\mathfrak{F},\quad \bar{f}\mapsto A_{+}^{*}\left(\bar{f}\right)|0\rangle\\ \Gamma_{-}:\mathcal{E}\ &\rightarrow\mathfrak{F},\quad f\mapsto A_{-}^{*}\left(f\right)|0\rangle\end{aligned}$$

then each $\Gamma_{\pm}$ *is a linear Hilbert space isomorphism onto the appropriately charged* 1*-particle subspace of* $\mathfrak{F}$,[7] *and*

$$\begin{aligned}\Gamma_{+}^{*}H\Gamma_{+}&=\bar{\Omega}\\ \Gamma_{-}^{*}H\Gamma_{-}&=\Omega\,.\end{aligned}$$

As promised, we now sketch an equivalent quantization which does not rely on a arbitrary choice of basis. Given a solution φ_{cl} to the classical equations of motion (20), define the (unquantized) linear functionals $\bar{A}_{+}$ and $\bar{A}_{-}$ on $\mathcal{E}^{*}$ and $\mathcal{E}$, respectively, by setting

$$\bar{A}_{+}\left(\bar{f}\right)=\frac{1}{\sqrt{2}}\int\varphi_{\mathrm{cl}}\left(t,x\right)\left(\bar{\Omega}^{1/2}e^{-it\bar{\Omega}}\bar{f}\right)(x)-i\frac{\partial\varphi_{\mathrm{cl}}\left(t,x\right)}{\partial t}\left(\bar{\Omega}^{-1/2}e^{-i\bar{\Omega}t}\bar{f}\right)(x)\;dx \tag{30}$$

$$\bar{A}_{-}\left(f\right)=\frac{1}{\sqrt{2}}\int\bar{\varphi}_{\mathrm{cl}}\left(t,x\right)\left(\Omega^{1/2}e^{-it\Omega}f\right)(x)-i\frac{\partial\bar{\varphi}_{\mathrm{cl}}\left(t,x\right)}{\partial t}\left(\Omega^{-1/2}e^{-i\Omega t}f\right)(x)\;dx \tag{31}$$

for $f\in\mathcal{E}$. Equations $(30)-(31)$ are independent of t, since φ_{cl} satisfies the Klein-Gordon equation, eq. (20). We then replace the $\bar{A}_{\pm}$ by operator-valued linear functionals $A_{\pm}^{*}$ satisfying $(22-23)$. The quantized field $\varphi_{RT}\left(t,\bar{f}\right)$ is then defined by equation (26). The Fock space is defined in the obvious way, and the Hamiltonian may be defined by

$$H=\sum_{k}A_{+}^{*}\left(\bar{\Omega}^{1/2}\bar{f}_{k}\right)A_{+}\left(\Omega^{1/2}f_{k}\right)+\sum_{k}A_{-}^{*}\left(\Omega^{1/2}f_{k}\right)A_{-}\left(\bar{\Omega}_{k}^{1/2}\bar{f}_{k}\right),$$

where $\{f_k\}$ is an arbitrary orthonormal basis of $\mathcal{E}$. The quantization of §1.2 may be recovered using the equations $\alpha_{+}^{*}\left(k\right)=A_{+}^{*}\left(\bar{e}_{k}\right)$, $\alpha_{-}^{*}\left(-k\right)=A_{-}^{*}\left(e_{k}\right)$.

[6]For $\mu>0$, equations $(26)-(27)$ show that if we restrict $\varphi_{RT}\left(t,\bar{f}\right)$ and $\varphi_{RT}^{*}\left(t,f\right)$ to the subspace of $\mathfrak{F}$ containing states of at most n particles, for fixed $n<\infty$, then we may continuously extend $\varphi_{RT}\left(t,\bar{f}\right)$ and $\varphi_{RT}^{*}\left(t,f\right)$ to $f\in\mathcal{E}^{-1/2}$, the completion of $\mathcal{E}$ in the inner product $\langle f,g\rangle_{-1/2}=\left\langle\Omega^{-1/2}f,\Omega^{-1/2}g\right\rangle$.

[7]Note that to any given linear Hilbert-space isomorphism $\tilde{\Gamma}_{+}:\mathcal{E}\rightarrow\mathfrak{F}_{+}^{(1)}=\mathrm{Span}\left\{\alpha_{+}^{*}\left(k\right)|0\rangle\right\}$ corresponds the antiunitary operator $f\mapsto\tilde{\Gamma}_{+}^{*}\Gamma_{+}\left(\bar{f}\right)$. Hence *a natural linear isomorphism between* $\mathcal{E}$ *and* $\mathfrak{F}_{1}^{+}$ *exists only if* $\mathcal{E}$ *is equipped with a preferred antiunitary operator.* Such is the case neither if $\mathcal{E}$ is produced by the Stone-von Neumann theorem from the (exponentiated) quantum-mechanical canonical commutation relations nor if $\mathcal{E}$ is the set of square-integrable sections of an arbitrary vector-bundle.

3 Implementing Lagrangian Symmetries on Fock Space

To motivate the definition of the Fock-space implementation of Lagrangian symmetries, we examine the adjoint substitution of test-functions which implements a unitary Lagrangian symmetry $S : \mathcal{E} \to \mathcal{E}$ at the classical level. If we replace $\varphi_{\rm cl} \to S\varphi_{\rm cl}$ then

$$\int \varphi_{\rm cl}(t,x)\, \bar{f}(x)\, dx \to \int S\varphi_{\rm cl}(t,x)\, \bar{f}(x)\, dx = \langle f(\cdot), S\varphi_{\rm cl}(t,\cdot)\rangle_{\mathcal{E}} = \int \varphi_{\rm cl}(t,x) \left(\bar{S}^* \bar{f}\right)(x)\, dx \tag{32}$$

where $\bar{S}^* \equiv \left(\bar{S}\right)^* = \overline{(S^*)}$. Similarly,

$$\int \bar{\varphi}_{\rm cl}(t,x)\, f(x)\, dx \to \int \left(\overline{S\varphi}_{\rm cl}\right)(t,x)\, f(x)\, dx = \int \bar{\varphi}_{\rm cl}(t,x)\, (S^* f)(x)\, dx. \tag{33}$$

The first and second transformations are implementable by the substitutions $\bar{f} \to \bar{S}^* \bar{f}$ and $f \to S^* f$, respectively. We use these adjoint substitutions (and similar considerations for antiunitary symmetries) as our definition:

Definition 9 *Let $S : \mathcal{E} \to \mathcal{E}$ be a Lagrangian symmetry. For S unitary, the corresponding **Fock-space implementation** $\mathbf{U}_S : \mathfrak{F} \to \mathfrak{F}$ is the linear operator which satisfies*

$$U_S |0\rangle = |0\rangle$$
$$U_S \varphi_{RT}\left(t, \bar{f}\right) U_S^* = \varphi_{RT}\left(t, \bar{S}^* \bar{f}\right) \tag{34}$$
$$U_S \varphi_{RT}^*(t, f)\, U_S^* = \varphi_{RT}^*(t, S^* f). \tag{35}$$

If S is an antiunitary symmetry then $\mathbf{U}_S : \mathfrak{F} \to \mathfrak{F}$ is given by

$$U_S |0\rangle = |0\rangle$$
$$U_S \varphi_{RT}\left(t, \bar{f}\right) U_S^* = \varphi_{RT}^*(t, S^* f)$$
$$U_S \varphi_{RT}^*(t, f)\, U_S^* = \varphi_{RT}\left(t, \bar{S}^* \bar{f}\right).$$

Note that for antiunitary S the adjoint S^* satisfies

$$\langle f, Sg\rangle = \overline{\langle S^* f, g\rangle}.$$

The following lemma gives the properties of U_S:

Lemma 10 *Let S and V be a unitary and an anti-unitary Lagrangian symmetries, respectively. Then U_S and U_V exist, are unitary, and commute with H. Furthermore, the actions of U_S and U_V on $\mathfrak{F}$ are given by*

$$U_S |0\rangle = |0\rangle$$
$$U_S A_+^*\left(\bar{f}\right) U_S^* = A_+^*\left(\bar{S}^* \bar{f}\right) \tag{36}$$
$$U_S A_-^*(f)\, U_S^* = A_-^*(S^* f) \tag{37}$$

and

$$U_V |0\rangle = |0\rangle$$
$$U_V A_+^*\left(\bar{f}\right) U_V^* = A_-^*(V^* f)$$
$$U_V A_-^*(f)\, U_V^* = A_+^*\left(\bar{V}^* \bar{f}\right).$$

PROOF. If U_S exists then it follows from $(30)-(31)$ that it satisfies (36)-(37). Existence and unitarity follow from the tensor product structure of $\mathfrak{F}$. The fact that U_S commutes with H follows from $(24)-(25)$.

We omit the similar proof of the antiunitary case. ■

3.1 TC Invariance

Theorem 11 *There is a unique antilinear operator TC on $\mathfrak{F}$ such that $TC\,|0\rangle = |0\rangle$ and*

$$\begin{aligned} TC\,\varphi_{RT}\left(t,\bar{f}\right)\,(TC)^{-1} &= \varphi^*_{RT}\left(-t,f\right) \\ TC\,\varphi^*_{RT}\left(t,f\right)\,(TC)^{-1} &= \varphi_{RT}\left(-t,\bar{f}\right). \end{aligned}$$

Furthermore, TC is antiunitary, squares to the identity, and satisfies

$$\begin{aligned} TC\,A^*_+\left(\bar{f}\right)\,(TC)^* &= A^*_-\left(f\right) \\ TC\,A^*_-\left(f\right)\,(TC)^* &= A^*_+\left(\bar{f}\right). \end{aligned}$$

The proof is similar to that of lemma 10 and is omitted. Notice that by the CCR $(18)-(19)$, there are no linear or antilinear operators $\tilde{T}$ and $\tilde{C}$ on $\mathfrak{F}$ with the properties that

$$\tilde{T}\varphi_{RT}\left(t,\bar{f}\right)\tilde{T}^{-1} = \varphi_{RT}\left(-t,\bar{f}\right) \quad , \quad \tilde{T}\varphi^*_{RT}\left(t,f\right)\tilde{T}^{-1} = \varphi^*_{RT}\left(-t,f\right)$$

and

$$\tilde{C}\varphi_{RT}\left(t,\bar{f}\right)\tilde{C}^{-1} = \varphi^*_{RT}\left(t,f\right) \quad , \quad \tilde{C}\varphi^*_{RT}\left(t,f\right)\tilde{C}^{-1} = \varphi_{RT}\left(t,\bar{f}\right).$$

However, if the Hilbert space $\mathcal{E}$ carries a conjugation[8] $^\vee$ which commutes with Ω, then it is easy to verify that there is an antiunitary operator $T^\vee$ and a unitary operator $C^\vee$ on $\mathfrak{F}$ such that

$$\begin{aligned} T^\vee|0\rangle = C^\vee|0\rangle &= |0\rangle \\ T^\vee\varphi_{RT}\left(t,\bar{f}\right)T^{\vee -1} &= \varphi_{RT}\left(-t,\overline{f^\vee}\right) \\ T^\vee\varphi^*_{RT}\left(t,f\right)T^{\vee -1} &= \varphi_{RT}\left(-t,f^\vee\right) \\ C^\vee\varphi_{RT}\left(t,\bar{f}\right)C^{\vee -1} &= \varphi^*_{RT}\left(t,f^\vee\right) \\ C^\vee\varphi^*_{RT}\left(t,f\right)C^{\vee -1} &= \varphi_{RT}\left(t,\overline{f^\vee}\right). \end{aligned}$$

Furthermore, $T^\vee C^\vee = TC$.

TC symmetry plays a crucial role in the proofs to follow. We summarize behavior of U_S and φ under TC symmetry:

Lemma 12 *Let $S : \mathcal{E} \to \mathcal{E}$ be a Lagrangian symmetry. Then*

1. $[U_S, TC] = 0$
2. $TC\,\varphi\left(t,\bar{f}\right)\,TC = \bar{\varphi}\left(t,f\right)$
3. $TC\,\bar{\varphi}\left(t,g\right)\,TC = \varphi\left(t,\bar{g}\right)$

[8] A conjugation is an antiunitary map that squares to 1.

Finally, we state the following Theorem, which shows that the class of Lagrangian symmetries contains most of the symmetries encountered in practice.

Theorem 13 *Let U be a unitary operator on the one-particle subspace $\mathfrak{F}^{(1)}$ of $\mathfrak{F}$ such that*

1. *U commutes with H and TC.*
2. *U maps $\mathfrak{F}^{(1)}_{\pm}$ either to itself or to $\mathfrak{F}^{(1)}_{\mp}$.*[9]

Then there exists a unique Lagrangian symmetry S such that

$$U = U_S|_{\mathfrak{F}^{(1)}}$$

The proof is a simple application of the isomorphisms $\Gamma_{\pm}$ of Theorem 8.

4 Twist Positivity

Having finished with our investigation of the Fock-space implementations of Lagrangian symmetries, we may now prove the anticipated theorems. That the partition function is well-defined follows from

Lemma 14 *If Ω is an admissible classical frequency operator (see definition 1) then $e^{-\beta H}$ is trace-class.*

Proof. Since $\omega_k > 0$,

$$\operatorname{Tr}\left(e^{-\beta H}\right) = \prod_k \left(\frac{1}{1-e^{-\beta\omega_k}}\right)^2 = \left(\prod_k \left(1 + \frac{e^{-\beta\omega_k}}{1-e^{-\beta\omega_k}}\right)\right)^2,$$

the conclusion follows from the estimate

$$\sum_k \frac{e^{-\beta\omega_k}}{1-e^{-\beta\omega_k}} \leq \frac{1}{1-e^{-\beta\mu}} \operatorname{Tr}\left(e^{-\beta\Omega}\right). \qquad \blacksquare$$

The next theorem shows that many symmetry operators are twist positive.

Theorem 15 *Let $S : \mathcal{E} \to \mathcal{E}$ be a linear or antilinear Lagrangian symmetry of an admissible free Lagrangian $\mathfrak{L}$. Then the Fock space implementation U_S of S is twist positive. Furthermore, for antiunitary S we have*

$$\operatorname{Tr}\left(U_S e^{-\beta H}\right) = \sqrt{\operatorname{Tr}\left(U_{S^2} e^{-2\beta H}\right)}. \tag{38}$$

We note that twist *nonnegativity* is a consequence TC symmetry (Lemma 12).
Proof. We first consider the case that S is unitary. Choosing the basis $\{e_k\}$ of section 2.1 to simultaneously diagonalize Ω and S,[10]

$$\Omega e_k = \omega_k e_k \tag{39}$$

$$S\, e_k = \rho_k e_k \tag{40}$$

[9] $\mathfrak{F}^{(1)}_{+}$ denotes the subspace of $\mathfrak{F}^{(1)}$ consisting of elements of the form $A^*_+(\bar{f})|0\rangle$. $\mathfrak{F}^{(1)}_{-}$ is defined analogously.

[10] Unitarity is used here, since an antiunitary operator is diagonalizable only if it is a conjugation.

we compute

$$\mathrm{Tr}_{\mathfrak{F}}\left(U_S e^{-\beta H}\right) = \prod_k \frac{1}{\left|1-\rho_k e^{-\beta\omega_k}\right|^2}. \tag{41}$$

Twist positivity follows, since

$$\prod_k \frac{1}{\left|1-\rho_k e^{-\beta\omega_k}\right|^2} \geq \left(\prod_k \frac{1}{1+e^{-\beta\omega_k}}\right)^2 = e^{-2\sum\limits_k \log\left(1+e^{-\beta\omega_k}\right)} \geq e^{-2\,\mathrm{Tr}\,e^{-\beta\Omega}} > 0.$$

Although in section 6.2 below we shall see that the previous proof may be altered to include the antiunitary case,[11] the suggestive formula (38) suffices. Let $\left\{\left|f_i^+\right\rangle\right\}$ and $\left\{\left|f_j^-\right\rangle\right\}$ be orthonormal bases of the charged subspaces $\mathfrak{F}^{(+)}$ and $\mathfrak{F}^{(-)}$ of $\mathfrak{F}$. Since $U_S e^{-\beta H}$ maps $\mathfrak{F}^{(\pm)}$ to $\mathfrak{F}^{(\mp)}$,

$$\begin{aligned}\operatorname*{Tr}_{\mathfrak{F}} U_S e^{-\beta H} &= \sum_{i,j} \left\langle f_i^+\right| U_S e^{-\beta H}\left|f_j^-\right\rangle \left\langle f_j^-\right| U_S e^{-\beta H}\left|f_i^+\right\rangle = \sum_i \left\langle f_i^+\right| U_{S^2} e^{-2\beta H}\left|f_i^+\right\rangle \\ &= \operatorname*{Tr}_{\mathfrak{F}^{(+)}} U_{S^2} e^{-2\beta H} = \operatorname*{Tr}_{\mathfrak{F}^{(-)}} U_{S^2} e^{-2\beta H}.\end{aligned}$$

But S^2 is unitary, so

$$\operatorname*{Tr}_{\mathfrak{F}^{(+)}} U_{S^2} e^{-2\beta H} = \prod_k \frac{1}{1-\rho_k e^{-2\beta\omega_k}}, \tag{42}$$

where $\{\rho_k, \omega_k\}$ are the joint eigenvalues (counting multiplicity) of $\left(S^2, \Omega\right)$. Since $\left[S, S^2\right] = 0$, the nonreal ρ_k come in conjugate pairs, so both sides of (42) are nonnegative. Hence

$$\operatorname*{Tr}_{\mathfrak{F}} U_S e^{-\beta H} = \sqrt{\operatorname*{Tr}_{\mathfrak{F}^{(+)}} U_{S^2} e^{-2\beta H} \times \operatorname*{Tr}_{\mathfrak{F}^{(-)}} U_{S^2} e^{-2\beta H}},$$

proving (38). ■

5 The Twisted Pair Correlation Function

We study the pair-correlation function, defined for unitary S in definition 5. The twisted pair correlation is often written in the suggestive notation

$$C\left(t, \bar{f}; s, g\right) = \int\limits_{X\times X} C\left(t, x; s, y\right) \bar{f}\left(x\right) g\left(y\right) dx\, dy, \tag{43}$$

where

$$C\left(t, x; s, y\right) = \frac{1}{Z_{U_S}} \operatorname*{Tr}_{\mathfrak{F}} \left[\left(\varphi\left(t, x\right) \bar{\varphi}\left(s, y\right)\right)_+ U_S e^{-\beta H}\right]. \tag{44}$$

Here $C\left(t, x; s, y\right)$ is not a function, but is only symbolic expression similar to the expression $\varphi_{RT}\left(t, x\right)$ introduced in §2.1. Note that the trace operation in (44) is always assumed to be interchanged with the integral in (43).

[11] One may also use the (conjugationless) structure theorem in [4].

5.1 The Integral Kernel $\mathbf{C}(t,x;s,y)$

We begin with a suggestive argument that

$$\int_0^\beta ds \int_X dy\, C(t,x;s,y)\, g(s,y) = \left(-D^2+\Omega_x^2\right)^{-1} g(t,x) \tag{45}$$

for smooth functions $g \in \mathcal{T}_\beta$ satisfying the periodic boundary conditions (14) for D. This calculation is justified in the remainder of §5.

The field $\bar{\varphi}$ satisfies the analog of the imaginary-time Klein-Gordon equation,

$$\left(-\partial_s^2+\bar{\Omega}_y^2\right)\bar{\varphi}(s,y) = 0. \tag{46}$$

Using this we get an equation for the imaginary-time Feynman Green's function,

$$\left(-\partial_s^2+\bar{\Omega}_y^2\right)\left(\varphi(t,x)\,\bar{\varphi}(s,y)\right)_+ = \delta_{t-s}\delta_{x,y}, \tag{47}$$

where $\delta_{x,y}$ is the Dirac measure

$$\int_{X\times X} \bar{f}(x)\, g(y)\, \delta_{x,y}\, dx\, dy = \langle f,g\rangle_{\mathcal{E}}\,.$$

Integrate by parts, interchange the trace and $\left(-\partial_s^2+\Omega_y^2\right)$, and an apply(47), to obtain

$$\begin{aligned}
&\int_0^\beta ds \int_X dy\, C(t,x;s,y)\left(-\partial_s^2+\Omega_y^2\right) g(s,y) \\
&= g(t,x) - \int_X dy\, C(t,x;s,y)\,\partial_s g(s,y)\Big|_{s=0}^{\beta} + \int_X dy\ \left(\partial_s C(t,x;s,y)\right) g(s,y)\Big|_{s=0}^{\beta}
\end{aligned} \tag{48}$$

Using the definitions of φ and U_S for S unitary, and by cyclicity of the trace,

$$\begin{aligned}
\int_X dy\, C(t,x;\beta,y)\,\partial_s g(\beta,y) &= \frac{1}{Z}\,\mathrm{Tr}\left[\,\varphi(t,x)\,\bar{\varphi}\left(\beta,\partial_s g(\beta,\cdot)\right) U_S e^{-\beta H}\right] \\
&= \frac{1}{Z}\,\mathrm{Tr}\left[\varphi(t,x)\, U_S e^{-\beta H}\bar{\varphi}\left(0,\partial_s S^* g(\beta,\cdot)\right)\right] \\
&= \frac{1}{Z}\,\mathrm{Tr}\left[\bar{\varphi}\left(0,\partial_s S^* g(\beta,\cdot)\right)\varphi(t,x)\, U_S e^{-\beta H}\right] \\
&= \int_X dy\, C(t,x;0,y)\,\partial_s S_y^* g(\beta,y)\,.
\end{aligned}$$

The second term in (48) vanishes by applying the boundary condition (14) on g. Similarly, the third term also vanishes, and hence

$$\int_0^\beta ds \int_X dy\, C(t,x;s,y)\left(-\partial_s^2+\Omega_y^2\right) g(s,y) = g(t,x)\,,$$

suggesting that

$$\int_0^\beta ds \int_X dy\, C(t,x;s,y)\, g(s,y) = \left(-\partial_t^2+\Omega_x^2\right)^{-1} g(t,x)\,,$$

as desired.

In the rest of this section, we make precise and justify the above manipulations.

5.2 Preliminary Estimates and Decomposition of C_β

We need an estimate to show that C_β is well-defined and bounded:

Lemma 16 *Let Ω be an admissible classical frequency operator, and let $\beta > 0$. Then for any $n \in \mathbb{Z}^+$, $t_1, ..., t_n \in [0, \beta]$, and $f_1, ..., f_n \in \mathcal{E}$ the time-ordered product*

$$\left(\varphi^\natural\left(t_1, f_1^\natural\right) ... \varphi^\natural\left(t_n, f_n^\natural\right)\right)_+ e^{-\beta H},$$

where the $\natural$'s stand for the independent presence or absence of a bar, extends to a unique trace-class operator. Furthermore, for each such n and β there exists a constant $K_{\beta,n}$ such that

$$\operatorname{tr}\left|\left(\varphi^\natural\left(t_1, f_1^\natural\right) ... \varphi^\natural\left(t_n, f_n^\natural\right)\right)_+ e^{-\beta H}\right| < K_{\beta,n} \prod_{i=1}^{n} \left\|\Omega^{-1/2} f_i\right\|_{\mathcal{E}} \tag{49}$$

for all $t_1, ..., t_n \in [0, \beta]$.

Proof. We note that $e^{-\alpha H}$ for $\alpha > 0$ maps $\mathfrak{F}$ into the domain of $\sqrt{N}$, which is contained in the any time-ordered product of imaginary-time fields. Hence expression (49) is certainly well-defined if all the t_k are less than β.

By equations $(24 - 25, 28 - 29)$ and the trace-norm Minkowski inequality, we need only consider terms of the form

$$f(a_1, ..., a_{n+1}) = e^{-a_1 H} A_\pm^\# \left(g_1^\natural\right) e^{-a_2 H} A_\pm^\# \left(g_2^\natural\right) ... A_\pm^\# \left(g_n^\natural\right) e^{-a_{n+1} H}, \tag{50}$$

where $g_i = \Omega^{-1/2} f_i$, $\sum_{i=1}^{n+1} a_i = \beta > 0$, each $a_i \geq 0$, and where $\#$ indicates the presence or absence of a $*$. For simplicity, we bound (50) in the case that all the $A_\pm^\#$ are $A_+^\#$.

Define the linear functionals $B^* : \mathcal{E}^* \to B(\mathfrak{F})$ and $B : \mathcal{E} \to B(\mathfrak{F})$ by

$$\begin{aligned} B^*(\bar{g}) &= A_+^*(\bar{g}) (N_+ + 1)^{-1/2} \\ B(g) &= (B^*(\bar{g}))^*, \end{aligned}$$

where $N_+ = \sum_k a_+^*(k) a_+(k)$. Then for any $g \in \mathcal{E}$ and any function $h : \mathbb{Z} \to \mathbb{C}$

$$\left\|B^\#\left(g^\natural\right)\right\|_{\mathfrak{F}} \leq \|g\|_{\mathcal{E}} \tag{51}$$

$$h(N_+) B^*(\bar{g}) = B^*(\bar{g}) h(N_+ + 1) \tag{52}$$

$$h(N_+ + 1) B(g) = B(g) h(N_+). \tag{53}$$

Temporarily fix the values of the a_i, and pick $a_j \geq \beta/(n+1)$. Consider equation (50) in terms of the B and B^* operators. Using $(52) - (53)$, to put the factors $\sqrt{N_+ + s}$ all next to $\exp(-a_j H)$, we have

$$f = e^{-a_1 H} B^\#\left(g_1^\natural\right) e^{-a_2 H} B^\#\left(g_2^\natural\right) ... \left(\sqrt{P(N_+)} e^{-a_j H/2}\right) e^{-a_j H/2} ... B^\#\left(g_n^\natural\right) e^{-a_{n+1} H} \tag{54}$$

where P is a degree-n polynomial satisfying

$$|P(x)| \leq (x + n + 1)^n \text{ for } x \geq 0 \tag{55}$$

From the inequality $H \geq \mu N_+$, we get

$$\left\| P\left(N_+\right) e^{-a_j H/2} \right\| \leq \sup_{x \geq 0} \left(x+n+1\right)^n e^{-\mu x} < \infty. \tag{56}$$

The existence and uniqueness of a bounded extension in the case $a_{n+1} = 0$ is now clear from (54). Then equation (54) expresses $f\left(a_1, ..., a_n\right)$ as a product of $e^{-a_j H/2}$ with many bounded operators. Applying equations (51), (54), and (56) and the choice of j gives

$$\operatorname{tr} \left| f\left(a_1, ..., a_{n+1}\right) \right| \leq \operatorname{tr} \left| e^{-\frac{\beta H}{2n+2}} \right| \times \left(\sup_{x \geq 0} \left(x+n+1\right)^n e^{-\mu x} \right) \prod_{i=1}^{n} \left\| \Omega^{-1/2} f_i \right\|.$$

But $\exp\left(-\beta H/\left(2n+2\right)\right)$ is trace-class by Lemma 14. The a_i were arbitrary, so (49) is proved. ■

We now have

Theorem 17 *$C_\beta : \mathcal{T}_\beta \to \mathcal{T}_\beta$ is well-defined, bounded, and self-adjoint.*

PROOF. Let $f, g : [0, \beta) \to \mathcal{E}$ be in $\mathcal{T}_\beta$. By Lemma 16 and Schwarz's inequality for $L^2\left(0, \beta\right)$,

$$\left| \int_0^\beta \int_0^\beta \operatorname{tr} \left(\varphi\left(t, \bar{f}\left(t\right)\right) \bar{\varphi}\left(s, g\left(s\right)\right) \right)_+ U_S e^{-\beta H} \, dt \, ds \right| \leq \beta K_{\beta,2} \left\| \Omega^{-1/2} f \right\|_{\mathcal{T}_\beta} \left\| \Omega^{-1/2} g \right\|_{\mathcal{T}_\beta}.$$

Hence C_β is well-defined, exists, and is bounded by the Riesz representation theorem. The self-adjointness of C_β is an immediate consequence of TC symmetry (Lemma 12). ■

C_β behaves nicely under direct sum decompositions:

Lemma 18 *Let Ω be a classical frequency operator of an admissible Lagrangian with a unitary Lagrangian symmetry S. Let the classical space $\mathcal{E}$ be decomposed into a direct sum of invariant subspaces of Ω and S,*

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus ... \qquad \Omega = \Omega_1 \oplus \Omega_2 \oplus ... \qquad S = S_1 \oplus S_2 \oplus ...$$

Let $\mathcal{T}_{\beta,j} = L^2\left[0, \beta\right) \otimes \mathcal{E}_j$. Then C_β also decomposes into a direct sum:

$$C_\beta = C_{\beta,1} \oplus C_{\beta,2} \oplus ...$$

where $C_{\beta,j} : \mathcal{T}_{\beta,j} \to \mathcal{T}_{\beta,j}$ is the S_j-twisted pair correlation operator of the free Bosonic theory with classical frequency operator Ω_j.

The proof is straightforward.

5.3 Rigorous Characterization of C_β

We need three more technical lemmas. The first concerns inverses of possibly unbounded self-adjoint operators:

Lemma 19 *Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$ with B bounded so that*

$$BA = 1|_{\mathfrak{D}(A)}.$$

Then B maps $\mathcal{H}$ into $\mathfrak{D}(A)$ and

$$AB = 1 = 1|_{\mathcal{H}}.$$

PROOF. We would like to say $(AB)^* = B^*A^* = BA = 1$, but since A may be unbounded we must be careful about domains. For $u \in \mathfrak{D}(A)$ and $x \in \mathcal{H}$,

$$u \mapsto \langle Au, Bx\rangle = \langle BAu, x\rangle = \langle u, x\rangle.$$

is a bounded function of u. Hence $Bx \in \mathfrak{D}(A^*) = \mathfrak{D}(A)$ and

$$\langle u, ABx\rangle = \langle u, x\rangle.$$

Since $\mathfrak{D}(A)$ is dense,

$$ABx = x,$$

and so $AB = 1$. ■

Definition 20 *Let $\mathcal{H}$ be a Hilbert space, and let X be a measure space. An operator-valued function $A : X \to B(\mathcal{H})$ is **weakly measurable** if the function $(v, f(x)\, w)$ is a measurable function of x for each $v, w \in \mathcal{H}$. The integral of such a function is defined by*

$$\left\langle v, \int A(x)\, w\, dx \right\rangle = \int \langle v, A(x)\, w\rangle\, dx$$

for all $v, w \in \mathcal{H}$.

Lemma 21 (Semi-noncommutative Fubini Theorem) *Let X be a measure space, $\mathcal{H}$ be a Hilbert space, and $A : X \to B(\mathcal{H})$ be a weakly measurable function. If*

$$\int_X \operatorname{Tr} |A(x)|\, dx < \infty$$

then

$$\int_X \operatorname{Tr} A(x)\, dx = \operatorname{Tr} \int_X A(x)\, dx. \tag{57}$$

PROOF. Let $\{e_k\}$ be an arbitrary basis of $\mathcal{H}$. Then the inequality

$$\int_X \sum_k |\langle e_k, A(x)\, e_k\rangle|\, dx \leq 4 \int_X \operatorname{Tr} |A(x)|\, dx$$

follows from the decomposition of $A(x)$ as a linear combination of positive operators, all of which have trace norm $\operatorname{Tr}|\cdot|$ less than or equal to $\operatorname{Tr}|A(x)|$:

$$A(x) = (\operatorname{Re} A(x))_+ + (\operatorname{Re} A(x))_- + i\,(\operatorname{Im} A(x))_+ + i\,(\operatorname{Im} A(x))_- .$$

Here $\operatorname{Re}(A) = \frac{1}{2}(A + A^*)$, $\operatorname{Im}(A) = \frac{1}{2i}(A - A^*)$, and $B_\pm = \frac{1}{2}(B \pm |B|)$. Equation (57) follows by Fubini's theorem, where the summation over k is considered to be an abstract Lebesgue integral in the counting measure. ∎

Lemma 22 *Let Ω be admissible and let $t \in [0, \beta)$. Then $\varphi\left(t, \bar{f}\right) \bar{\varphi}(\beta, g)\, U_S e^{-\beta H}$ has a unique bounded extension, which is trace-class and satisfies*

$$\operatorname{tr}\left(\varphi\left(t, \bar{f}\right) \bar{\varphi}(\beta, g)\, U_S e^{-\beta H}\right) = \operatorname{tr}\left(\bar{\varphi}(0, S^* g)\, \varphi\left(t, \bar{f}\right) U_S e^{-\beta H}\right).$$

PROOF. By Lemma 16, $\varphi\left(t, \bar{f}\right) \bar{\varphi}(\beta, g)\, U_S e^{-\beta H}$ has a unique bounded extension, which is trace-class. Writing

$$\varphi\left(t, \bar{f}\right) \bar{\varphi}(\beta, g)\, U_S e^{-\beta H} = \varphi\left(t, \bar{f}\right) U_S e^{-\beta H/2} \times e^{-\beta H/2} \bar{\varphi}(0, S^* g),$$

we notice that both factors extend to trace-class operators by Lemma 16. By a double-application of cyclicity of the trace,

$$\begin{aligned} \operatorname{tr}\left(\varphi\left(t, \bar{f}\right) \bar{\varphi}(\beta, g)\, U_S e^{-\beta H}\right) &= \operatorname{tr}\left(e^{-\beta H/2} \bar{\varphi}(0, S^* g)\, \varphi\left(t, \bar{f}\right) U_S e^{-\beta H/2}\right) \\ &= \operatorname{tr}\left(\bar{\varphi}(0, S^* g)\, \varphi\left(t, \bar{f}\right) U_S e^{-\beta H}\right). \end{aligned}$$ ∎

We may now make rigorous the argument of section 5.1.

Theorem 23 *Let S be a unitary Lagrangian symmetry of an admissible free Lagrangian $\mathcal{L}$. Then $C_\beta = \left(-D^2 + \Omega^2\right)^{-1}_{\mathcal{T}_\beta}$, where Ω is identified with $1 \otimes \Omega : \mathcal{T}_\beta \to \mathcal{T}_\beta$.*

PROOF. We claim that we only need consider the case that $\mathcal{E} = \mathbb{C}$. Since $[S, \Omega] = 0$, we may choose a basis $\{e_k\}$ of $\mathcal{E}$ of simultaneous eigenvectors of S and Ω. Then $\left(-D^2 + \Omega^2\right)^{-1}$ is reduced by the direct sum

$$\mathcal{T}_\beta = \oplus_k L^2[0, \beta) \otimes \operatorname{Span}(e_k).$$

By Lemma 18, C_β is also reduced, proving our claim.

By Lemma 19, all we have to show is that

$$\left\langle f, C_\beta \left(-D^2 + \Omega^2\right) g\right\rangle_{\mathcal{T}_\beta} = \langle f, g\rangle_{\mathcal{T}_\beta} \tag{58}$$

for g in the domain of $-D^2 + \Omega^2$. By standard Sobolev space results (or Lebesgue's density theorem), such g may be represented by a function which is twice-differentiable almost everywhere and satisfies

$$\begin{aligned} g(\beta) &= S g(0) \\ g'(\beta) &= S g'(0) \\ g'(b) - g'(a) &= \int_a^b g''(x)\, dx, \quad 0 \leq a \leq b \leq \beta, \end{aligned} \tag{59}$$

where S is now just a complex number and $Dg = g'$. For $E, F \in \mathfrak{F}_0$, we have the identity

$$\langle E|\ \bar{\varphi}\left(s, \left(-\frac{d^2}{ds^2} + \Omega^2\right) g(s)\right)\ |F\rangle = \frac{d}{ds}\langle E|\left(-\bar{\varphi}(s, g'(s)) + \frac{\partial \bar{\varphi}}{\partial s}(s, g(s))\right)|F\rangle.$$

Let $\{E_n\} \subseteq \mathfrak{F}$ be a basis of eigenfunctions of N. We compute

$$\begin{aligned}
&\left(f, C_\beta\ \left(-D^2 + \Omega^2\right) g\right) \\
&= \frac{1}{Z}\sum_n \int_0^\beta \int_0^t dt\, ds\ \frac{d}{ds}\langle E_n|\left(-\bar{\varphi}\left(s, \frac{dg}{ds}\right) + \frac{\partial \bar{\varphi}}{\partial s}(s, g)\right)\varphi\left(t, \bar{f}\ (t)\right) U_S e^{-\beta H}|E_n\rangle \\
&\quad + \frac{1}{Z}\sum_n \int_0^\beta \int_t^\beta dt\, ds\ \frac{d}{ds}\langle E_n|\,\varphi\left(t, \bar{f}\,(t)\right)\left(-\bar{\varphi}\left(s, \frac{dg}{ds}\right) + \frac{\partial \bar{\varphi}}{\partial s}(s, g)\right) U_S e^{-\beta H}|E_n\rangle \\
&= \frac{1}{Z}\sum_n \int_0^\beta dt\ \ \langle E_n|\left[-\bar{\varphi}\left(t, \frac{dg}{dt}\right) + \frac{\partial \bar{\varphi}}{\partial t}(t, g), \varphi\left(t, \bar{f}\,(t)\right)\right] U_S e^{-\beta H}|E_n\rangle + BT \qquad (60) \\
&= \frac{1}{Z}\sum_n \int_0^\beta dt\ \ \left(f(t), g(t)\right)_{\mathcal{E}}\ \langle E_n|\, U_S e^{-\beta H}|E_n\rangle + BT \\
&= (f, g)_{\mathcal{T}_\beta} + BT,
\end{aligned}$$

where BT stands for the boundary terms. We were able to move the integrations inside of the trace using Lemma 21 and the estimate of Lemma 16. Equation (60) used (59).

We consider the boundary terms:

$$\begin{aligned}
BT &= -\frac{1}{Z}\sum_n \int_0^\beta dt\ \ \langle E_n|\left(-\bar{\varphi}(0, D_s g) + \frac{\partial \bar{\varphi}}{\partial s}(0, g)\right)\varphi\left(t, \bar{f}\,(t)\right) U_S e^{-\beta H}|E_n\rangle \\
&\quad + \frac{1}{Z}\sum_n \int_0^\beta dt\ \ \langle E_n|\,\varphi\left(t, \bar{f}\,(t)\right)\left(-\bar{\varphi}(\beta, D_s g) + \frac{\partial \bar{\varphi}}{\partial s}(\beta, g)\right) U_S e^{-\beta H}|E_n\rangle \\
&= \frac{1}{Z}\int_0^\beta dt\ \ \mathrm{Tr}\,\bar{\varphi}(0, D_s g(0))\,\varphi\left(t, \bar{f}\,(t)\right) U_S e^{-\beta H} \\
&\quad - \frac{1}{Z}\int_0^\beta dt\ \ \mathrm{Tr}\,\varphi\left(t, \bar{f}\,(t)\right)\bar{\varphi}(\beta, D_s g(\beta))\, U_S e^{-\beta H} \\
&\quad + \frac{1}{Z}\int_0^\beta dt\ \ \mathrm{Tr}\,\varphi\left(t, \bar{f}\,(t)\right)\frac{\partial \bar{\varphi}}{\partial s}(\beta, g(\beta))\, U_S e^{-\beta H} \\
&\quad - \frac{1}{Z}\int_0^\beta dt\ \ \mathrm{Tr}\,\frac{\partial \bar{\varphi}}{\partial s}(0, g(0))\,\varphi\left(t, \bar{f}\,(t)\right) U_S e^{-\beta H}
\end{aligned}$$

We were able to interchange integration and the trace for the same reasons as above. The first two terms cancel by Lemma 22. The last two cancel similarly. ■

6 The Antiunitary Case, Real Fields

We would like to prove an analog of Theorem 23 for antiunitary classical symmetries, as well as a theorem for symmetries of real scalar fields. Given that we have not required the choice of an arbitrary conjugation on our classical space $\mathcal{E}$,[12] it is surprising that unification of the unitary and antiunitary cases results from consideration of the real scalar field.

6.1 The Extended Pair Correlation Operator

We note that if $V : \mathcal{E} \to \mathcal{E}$ is antiunitary then in general

$$\operatorname{tr}\left[\left(\varphi\left(t,x\right)\varphi\left(s,y\right)\right)_{+} U_V e^{-\beta H}\right] \neq 0.$$

Hence the important operator for Wick's theorem is no longer the pair correlation operator C_β. We define

Definition 24 *Let Ω be a classical frequency operator on $\mathcal{E}$ with antiunitary classical symmetry V. We define the* ***extended space of classical fields*** $\mathbb{E} = \mathcal{E}^* \oplus \mathcal{E}$. *The* ***extended path space*** *is*

$$\mathbb{T}_\beta = L^2\left(0,\beta\right) \otimes \mathbb{E},$$

and the ***extended twisted pair correlation operator*** $\hat{C}_\beta : \mathbb{T}_\beta \to \mathbb{T}_\beta$ *is the operator which satisfies*

$$\begin{aligned}
\left(\bar{f}\left(t\right) \oplus g\left(t\right), \hat{C}\, \bar{h}\left(t\right) \oplus k\left(t\right)\right)_{\mathbb{T}_\beta} = &\frac{1}{Z_{\beta,U_V}} \int\int \operatorname{tr}\left(\left(\bar{\varphi}\left(t, f\left(t\right)\right) \varphi\left(s, \bar{h}\left(s\right)\right)\right)_{+} U_V e^{-\beta H}\right)\, dt\, ds \\
&+ \frac{1}{Z_{\beta,U_V}} \int\int \operatorname{tr}\left(\left(\bar{\varphi}\left(t, f\left(t\right)\right) \bar{\varphi}\left(s, k\left(s\right)\right)\right)_{+} U_V e^{-\beta H}\right)\, dt\, ds \\
&+ \frac{1}{Z_{\beta,U_V}} \int\int \operatorname{tr}\left(\left(\varphi\left(t, \bar{g}\left(t\right)\right) \varphi\left(s, \bar{h}\left(s\right)\right)\right)_{+} U_V e^{-\beta H}\right)\, dt\, ds \\
&+ \frac{1}{Z_{\beta,U_V}} \int\int \operatorname{tr}\left(\left(\varphi\left(t, \bar{g}\left(t\right)\right) \bar{\varphi}\left(s, k\left(s\right)\right)\right)_{+} U_V e^{-\beta H}\right)\, dt\, ds.
\end{aligned}$$

We note that if the symmetry V were unitary, then the middle two terms would vanish, reducing consideration to the pair correlation operators associated to (Ω, V) and $\left(\bar{\Omega}, \bar{V}\right)$.

6.2 The relationship between real and complex scalar fields

We reduce consideration of antiunitary symmetries of a complex scalar field to consideration of (classically unitary) symmetries of a real scalar field. Had we required that our space of classical fields $\mathcal{E}$ come equipped with a conjugation which commuted with Ω[13] then our complex field theory would be a direct sum of two real fields.[14] Although we impose no reality condition on Ω nor conjugation on $\mathcal{E}$, we will see that in a certain sense our complex field *is* a real field.

[12] All use of the arbitary representation $\mathcal{E} = L^2\left(X\right)$ was for notational purposes only.

[13] so that the Klein-Gordon equation has real solutions

[14] We would likewise need to restrict consideration Lagrangian symmetries which commute with conjugation on $\mathcal{E}$.

Definition 25 *The* ***natural conjugation*** $\bar{\cdot} : \mathbb{E} \to \mathbb{E}$ *is given by*

$$\overline{\bar{f} \oplus g} = \bar{g} \oplus f.$$

A ***real operator*** $R : \mathbb{E} \to \mathbb{E}$ *is one that commutes with conjugation. Let* $\Omega : \mathcal{E} \to \mathcal{E}$ *be a classical frequency operator of a complex field* φ_{RT}*, as above. Let* S *and* V *be unitary and antiunitary Lagrangian symmetries of* Ω*, respectively. Define the* ***associated real field*** $\psi_{RT} : \mathbb{R} \times \mathbb{E} \to Op(\mathfrak{F})$ *by*

$$\psi_{RT}\left(t, \bar{f} \oplus g\right) = \varphi_{RT}\left(t, \bar{f}\right) + \varphi_{RT}^{*}\left(t, g\right),$$

and the ***associated imaginary-time real field*** $\psi : \mathbb{R} \times \mathbb{E} \to Op(\mathfrak{F})$ *by*

$$\psi\left(t, \bar{f} \oplus g\right) = e^{-tH} \psi_{RT}\left(0, \bar{f} \oplus g\right) e^{tH}.$$

Furthermore, define

$$\begin{aligned}
\Omega_{\mathbb{R}} &= \bar{\Omega} \oplus \Omega \\
\mathbb{S} &= \bar{S} \oplus S \\
\mathbb{V}\left(\bar{f} \oplus g\right) &= \overline{Vg} \oplus Vf \\
\mathbb{A}^{*}\left(\bar{f} \oplus g\right) &= A_{+}^{*}\left(\bar{f}\right) + A_{-}^{*}\left(g\right) \\
\mathbb{A}\left(\bar{f} \oplus g\right) &= A_{-}\left(\bar{f}\right) + A_{+}\left(g\right) = \left(\mathbb{A}^{*}\left(\overline{\bar{f} \oplus g}\right)\right)^{*} \\
\mathbb{T}_{\beta} &= L^{2}\left[0, \beta\right) \otimes \mathbb{E}
\end{aligned}$$

and define $\mathbb{D}_{\mathbb{S}}$ *and* $\mathbb{D}_{\mathbb{V}} : \mathbb{T}_{\beta} \to \mathbb{T}_{\beta}$ *analogously to* D*.*

We have the following

Theorem 26 ψ_{RT} *is a free real scalar field with classical frequency operator* $\Omega_{\mathbb{R}}$*, i.e.*

1. $\Omega_{\mathbb{R}}$ *is real.*
2. ψ_{RT} *is self-adjoint, i.e.* $\left(\psi_{RT}\left(t, q\right)\right)^{*} = \psi_{RT}\left(t, \bar{q}\right)$*.*
3. $\partial_{t}^{2} \psi_{RT}\left(t, q\right) = -\psi_{RT}\left(t, \Omega_{\mathbb{R}}^{2} q\right)$ *strongly on* $\mathfrak{F}_{0}$ *for* $q \in \mathcal{D}\left(\Omega_{\mathbb{R}}^{2}\right)$*.*
4. $\left[\psi_{RT}\left(t, q\right), \frac{\partial \psi_{RT}}{\partial t}\left(t, r\right)\right] = i\left(\bar{q}, r\right)_{\mathbb{E}}$*.*
5. $\left[\psi_{RT}\left(t, q\right), \psi_{RT}\left(t, r\right)\right] = 0$*.*
6. *Successively applying* $\psi_{RT}\left(t, \cdot\right)$ *and* $\partial_{t} \psi_{RT}\left(t, \cdot\right)$*, and to* $\left|0\right\rangle$ *gives a dense subset of* $\mathfrak{F}$*.*

   Furthermore,

7. $\mathbb{A}^{*}$ *is the creation functional of* ψ_{RT}*, i.e.*

$$\psi_{RT}\left(t, q\right) = \frac{1}{\sqrt{2}}\left(\mathbb{A}^{*}\left(\Omega_{\mathbb{R}}^{-1/2} e^{it\Omega_{\mathbb{R}}} q\right) + \mathbb{A}\left(\Omega_{\mathbb{R}}^{-1/2} e^{-it\Omega_{\mathbb{R}}} q\right)\right)$$

   and

$$\left[\mathbb{A}\left(q\right), \mathbb{A}^{*}\left(r\right)\right] = \left\langle \bar{q}, r \right\rangle_{\mathbb{E}}.$$

8. $\mathbb{S}$ *and* $\mathbb{V}$ *are real and unitary, and the real-field Fock space implementations of* $\mathbb{S}$ *and* $\mathbb{V}$*, which satisfy*

$$\begin{aligned}\mathbb{U}_{\mathbb{S}}\left|0\right\rangle &= \mathbb{U}_{\mathbb{V}}\left|0\right\rangle = \left|0\right\rangle \\ \mathbb{U}_{\mathbb{S}}\psi_{RT}\left(t,q\right)\mathbb{U}_{\mathbb{S}}^{*} &= \psi_{RT}\left(t,\mathbb{S}^{*}q\right) \\ \mathbb{U}_{\mathbb{V}}\psi_{RT}\left(t,q\right)\mathbb{U}_{\mathbb{V}}^{*} &= \psi_{RT}\left(t,\mathbb{V}^{*}q\right),\end{aligned}$$

are simply given by

$$\mathbb{U}_{\mathbb{S}} = U_S \quad \textit{and} \quad \mathbb{U}_{\mathbb{V}} = U_V.$$

9. $\hat{C}$ *is the twisted pair correlation operator of* ψ_{RT} *with the symmetry* $\mathbb{V}$*, i.e.*

$$\left(\bar{f}\oplus g,\hat{C}\,\bar{h}\oplus k\right) = \int_0^\beta\int_0^\beta \operatorname{tr}\left[\left(\psi\left(t,\overline{\bar{f}\oplus g}\right)\psi\left(s,\bar{h}\oplus k\right)\right)_+ U_V e^{-\beta H}\right]\,dt\,ds.$$

We have the following theorem concerning real scalar fields:

Theorem 27 *Let* $\tilde{\mathcal{E}}$ *be a Hilbert space with conjugation, and let* $\varphi_{RT}^{\mathbb{R}}:\mathbb{R}\times\tilde{\mathcal{E}}\to\mathbb{C}$ *be a free real scalar field with admissible real classical frequency operator* $\tilde{\Omega}:\tilde{\mathcal{E}}\to\tilde{\mathcal{E}}$*. Let* $\tilde{S}$ *be a real unitary Lagrangian symmetry of* $\tilde{\Omega}$*. Then* $U_{\tilde{S}}$ *is twist-positive and the corresponding pair-correlation operator* $\tilde{C}_\beta$ *satisfies*

$$\tilde{C}_\beta = \left(-\tilde{D}^2+\tilde{\Omega}^2\right)_{\tilde{T}_\beta}^{-1}, \tag{61}$$

where $\tilde{D}$ *is defined analogously to* D*, and where* $\tilde{\Omega}$ *is identified with* $I\otimes\tilde{\Omega}:\tilde{T}_\beta\to\tilde{T}_\beta$*.*

Proof. Twist positivity is proved by replacing the use of TC symmetry in the proof of Theorem 15 with the observation that the nonreal eigenvalues of the real operator $\tilde{S}$ come in complex conjugate pairs. Equation (61) may be proved by slight notational changes in the proof of Theorem 23. ■

The previous two theorems reduce the antiunitary case to a triviality:

Corollary 28 *Let* Ω *be an admissible classical frequency operator of a free complex scalar field. Let* V *be an antiunitary Lagrangian symmetry. Then the extended pair correlation operator* $\hat{C}_\beta$ *is positive definite. In particular,*

$$\hat{C}_\beta = \left(-\mathbb{D}_{\mathbb{V}}^2+\Omega_{\mathbb{R}}^2\right)_{\mathbb{T}_\beta}^{-1},$$

where $\Omega_{\mathbb{R}}$ *is identified with* $I\otimes\Omega_{\mathbb{R}}$*.*

Furthermore, we note that Theorem 27 applies not only applies to Lagrangian symmetries of complex fields, but in general to symmetries which mix the subspaces $\mathcal{E}$ and $\mathcal{E}^*$ of $\mathbb{E}$.[15] Since the Fock-space implementations of these additional symmetries will mix particle and antiparticle states, they are somewhat less natural.

[15]The simplest example is given by $\mathcal{E}=\mathbb{C}^2$, $\tilde{S}\left(\bar{f}\oplus g\right) = \left[\overline{\left(\sigma_x f+s_z g\right)}\oplus\left(s_z f+\sigma_z g\right)\right]/\sqrt{2}$, where $\sigma_x\left(z_1,z_2\right)=\left(z_2,z_1\right)$, $s_z\left(z_1,z_2\right)=\left(\bar{z}_1,-\bar{z}_2\right)$, $f\mapsto\bar{f}$ is given by definition 1, and $z\mapsto\bar{z}$ is just complex conjugation.

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